Optimal guaranteed cost sliding mode control for a missile with unmatched uncertainties

Donghyeok Hwang[1] and Min-Jea Tahk[2]

Abstract If the matching condition of uncertainties is not satisfied, robustness of sliding mode control becomes limited. In order to solve the unmatched problem, this paper presents a sliding mode controller with a combination of an optimal guaranteed cost controller. The model uncertainties are assumed to be norm-bounded, but the matched condition needs not to be satisfied. The proposed sliding mode controller is designed to provide system invariance to disturbances and parameter variations with known bounds which are implicit in the control channel called matched uncertainties. In addition, the combined optimal guaranteed cost controller ensures the sliding motion to be robust to unmatched uncertainties by providing an upper bound on a given performance cost. As a result, system performance degradation incurred by unmatched uncertainties is guaranteed to be less than the given bound otherwise the sliding motion may become unstable and/or performance may degrade. Conditions for the existence of guaranteed cost sliding mode satisfying the given constraints are derived. The performance of the proposed schemes is illustrated by numerical simulations.

1. Introduction

The classical autopilot has been successfully employed as the design topology of choice during the past several years [1—3]. However, if there exist large model mismatches between design models and actual systems, the classical autopilot does not assure effective control. To improve the robustness of the classical three-loop topology, a "neoclassic" four-loop autopilot which uses four gains instead of three has been presented in [2]. In reference [3], ten distinct topologies that use combinations of acceleration, angular rate, and first order leads have been examined to determine the best from a robustness perspective. However, their robust performance cannot be guaranteed without exact modeling of the uncertainty.

Recently, sliding mode controllers which are designed for uncertainties and time-delay have drawn more and more attention, for example, autopilot design of agile missiles [4], autopilot design of aircraft [5], an integrated attitude and accel-

---

Donghyeok Hwang, Agency for Defense Development, dhhwang@add.re.kr

Min-Jea Tahk, Department of Aerospace Engineering, KAIST, mjtahk@kaist.ac.kr

eration controller for skid to turn missiles [6], and integrated guidance and control [7], and so on. However, if the matching condition of uncertainty (i.e., the uncertainty lies in the same channel as the input) is not satisfied, robustness of sliding mode control becomes limited [16]. Effort has been directed towards finding a controller in order to guarantee robust stability under the unmatched condition. Coordinate transformations have been used to solve the unmatched problem [8—9], but those approaches cause another difficulty in flight control system, because dynamics describing missile acceleration are non-minimum phase.

When controlling real plants, it is desirable to design controllers which not only make the closed-loop systems asymptotically stable but also guarantee an adequate level of performance. One approach to this problem is the guaranteed cost control approach [11—15]. Ricatti equation approaches for designing quadratic guaranteed cost controllers are presented in [11—13], and linear matrix inequality approaches for the design of guaranteed cost controller are formulated as convex optimization problem in [14—15].

On the basis of these literatures, this paper presents a sliding mode controller with a combination of an optimal guaranteed cost controller. In order to solve the unmatched problem, the proposed sliding mode controller is designed to provide system invariance to disturbances and parameter variations with known bounds which are implicit in the control channel called matched uncertainties. In addition, the combined guaranteed cost control ensures sliding motion to be robust to unmatched uncertainties by providing an upper bound on a given performance cost. As a result, system performance degradation incurred by unmatched uncertainties is guaranteed to be less than the given bound otherwise the sliding motion may become unstable and/or performance may degrade. Conditions for existence of the guaranteed cost sliding mode satisfying the given constraints are derived. The performances of the proposed scheme are illustrated by numerical simulations in the presence of unmatched model uncertainties.

2. Missile dynamics model with uncertainties

The longitudinal missile dynamics, using a small angle approximation, are given as

$$\dot{\alpha} = Z_1\alpha + q + Z_2\delta$$

$$\dot{q} = M_1\alpha + M_2\delta \tag{1}$$

The variable that is to be commanded is denoted as A_L, and modeled as

$$A_L = -VZ_1\alpha - VZ_2\delta \tag{2}$$

where α is the angle of attack, q is the pitch rate, V is the missile velocity, δ is the fin deflection, A_L is the missile normal acceleration, and Z_1, Z_2, M_1, and M_2 are the aerodynamics coefficients. Since the typical measurements available from the inertial measurement unit are normal acceleration A_L and pitch rate q, it is desirable to replace the angle of attack in system state description with normal acceleration as follows:

$$\begin{bmatrix} \dot{A}_L \\ \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} Z_1 & -VZ_1 & 0 \\ 0 & 0 & 1 \\ -\frac{M_1}{V} & M_1 & 0 \end{bmatrix} \begin{bmatrix} A_L \\ q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} -VZ_2 \\ 0 \\ M_2 \end{bmatrix} \dot{\delta} \tag{3}$$

The aerodynamics coefficients described by Z_1, Z_2, M_1, and M_2 are dependent on Mach number and angle-of-attack. These parameters are usually measured from wind tunnel tests and these values may contain some error because of the imperfection of the measurements. These errors are assumed to be bounded and can be modeled as multiplicative uncertainties

$$Z_i = (1 + \Delta_i)\bar{Z}_i$$

$$M_i = (1 + \Delta_i)\bar{M}_i \tag{4}$$

where $\bar{Z}_i$ and $\bar{M}_i$ are the estimated nominal values of the aerodynamic coefficients and Δ_i represent the admissible uncertainties, and their values are assumed to have a norm-bounded

$$\|\Delta_i\|_\infty < \frac{1}{\mu_i} \tag{5}$$

and can be expressed as normalized forms

$$\Delta_i = \frac{1}{\mu_i}\underline{\Delta}_i \tag{6}$$

where $\|\underline{\Delta}_i\|_\infty < 1$. In addition to uncertainty, time-delay is also a major source of instability and poor performance in actual applications. Thus, for a flight control system, a first order lag is considered that capture un-modeled dynamics which is reflected to the input of the missile dynamics. In other words, let

$$\ddot{\delta} = -\frac{1}{T}\dot{\delta} + \frac{1}{T}\dot{\delta}_c \tag{7}$$

where $\dot{\delta}$ is the fin rate, $\dot{\delta}_c$ is the fin rate command, and T is an uncertain delay with known bounds $0 < T_{min} \leq T \leq T_{max}$. Under these conditions, the longitudinal missile dynamic model with uncertainties can be represented in state space form as follows:

$$\begin{bmatrix} \dot{A}_L \\ \dot{q} \\ \ddot{q} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \bar{Z}_1(1+\Delta_1) & -V\bar{Z}_1(1+\Delta_2) & 0 & -V\bar{Z}_2(1+\Delta_5) \\ 0 & 0 & 1 & 0 \\ -\dfrac{\bar{M}_1(1+\Delta_3)}{V} & \bar{M}_1(1+\Delta_4) & 0 & \bar{M}_2(1+\Delta_6) \\ 0 & 0 & 0 & -\dfrac{1}{T} \end{bmatrix} \begin{bmatrix} A_L \\ q \\ \dot{q} \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{T} \end{bmatrix} \dot{\delta}_c \quad (8a)$$

or, with a simple expression,

$$\begin{bmatrix} \dot{x} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ 0 & -\dfrac{1}{T} \end{bmatrix} \begin{bmatrix} x \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{T} \end{bmatrix} u \qquad (8b)$$

Unfortunately, all the multiplicative uncertainties in Eq. (8a) are not implicit in the control channel, so all the effect of the uncertainties cannot be eliminated by sliding mode controller alone. In order to solve the unmatched problem, an optimal guaranteed cost sliding mode controller is presented and will be highlighted in the next section.

3. Optimal Guaranteed Cost Sliding Mode Control

3.1 Sliding surface design with virtual control

Let the sliding surface be

$$s = [[k_1 \quad k_2 \quad k_3] \quad 1] \begin{bmatrix} \begin{bmatrix} A_L \\ q \\ \dot{q} \end{bmatrix} \\ \dot{\delta} \end{bmatrix} - k_4 A_{LC} = [K \quad 1] \begin{bmatrix} x \\ u_{vir} \end{bmatrix} - k_4 A_{LC} \quad (9)$$

where $k_{1\sim3}$ coefficients are selected to achieve the characteristic required from the system when the state variables are in sliding mode. k_4 coefficient is computed so that the achieved acceleration will match the commanded acceleration. A_{LC} is the normal acceleration command, and A_L is commanded output. If we regard $\dot{\delta}$ in Eq. (9) as a virtual control input u_{vir} and try to design $\dot{\delta}_c$ in Eq. (8b) to bring the sliding variable s to zero in finite time and then maintain the condition $s = 0$ for all future time, the system characteristics are governed by the virtual control

$$u_{vir} = -Kx + k_4 A_{LC} \tag{10}$$

The sliding surface in Eq. (9) has relative degree one because the first time derivative of the sliding variable s is a function of control $\dot{\delta}_c$

$$\dot{s} = \sum_{i=1}^{3}\left(f_i + \frac{k_i}{T}\right)x_i + \sum_{i=4}^{5}\left(f_i - \frac{k_i}{T}\right)x_i + \frac{1}{T}\dot{\delta}_c \tag{11}$$

where $x_1 = A_L,\ x_2 = q,\ x_3 = \dot{q},\ x_4 = A_{LC},\ x_5 = s$ and $k_5 = 1$

$$f_1 = k_1 Z_1 - \frac{k_3 M_1}{V} + k_1^2 V Z_2 - k_1 k_3 M_2$$

$$f_2 = -k_1 V Z_1 + k_3 M_1 + k_1 k_2 V Z_2 - k_2 k_3 M_2$$

$$f_3 = k_2 + k_1 k_3 V Z_2 - k_3^2 M_2$$

$$f_4 = -k_1 k_4 V Z_2 + k_3 k_4 M_2$$

$$f_5 = -k_1 V Z_2 + k_3 M_2$$

Once sliding mode is established, the state variables satisfy the condition of $\dot{s} = 0$ and the equivalent control is given by

$$\dot{\delta}_{c_{eq}} = u_{vir} - TK\dot{x} \tag{12}$$

Substituting the equivalent control into Eq. (8b) yields the sliding mode equation as

$$\dot{x} = \tilde{A}x + \tilde{B}_1 u_{vir} \tag{13}$$

It is evident from Eq. (13) that once the sliding mode is established, the uncertain delay T is rejected from the sliding mode equation, and the flight control system becomes invariant to the effect of the uncertain delay. In order to design the guaranteed cost virtual control u_{vir}, Eq. (13) is transformed into the uncertainty pulled out forms, which can be rewritten as

$$\dot{x} = Ax + B_1 u_{vir} + B_2 w \tag{14}$$

$$z = Cx + Du_{vir}$$

$$w = \underline{\Delta} z$$

with

$$A=\begin{bmatrix}\bar{Z}_1 & -V\bar{Z}_1 & 0\\ 0 & 0 & 1\\ -\frac{\bar{M}_1}{V} & \bar{M}_1 & 0\end{bmatrix},\qquad B_1=\begin{bmatrix}-V\bar{Z}_2\\ 0\\ \bar{M}_2\end{bmatrix},\qquad B_2=\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{bmatrix}$$

$$C=\begin{bmatrix}\bar{Z}_1\frac{1}{g_1} & 0 & 0\\ 0 & -V\bar{Z}_1\frac{1}{g_2} & 0\\ -\frac{\bar{M}_1}{V}\frac{1}{g_3} & 0 & 0\\ 0 & \bar{M}_1\frac{1}{g_4} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad D=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ -V\bar{Z}_2\frac{1}{g_5}\\ \bar{M}_2\frac{1}{g_6}\end{bmatrix}$$

where $C$and D are known constant matrices and $\underline{\Delta}$ is unknown matrix satisfying $\underline{\Delta}^T\underline{\Delta} < I$. Note that μ_i are known bounds of uncertainties defined in Eq. (5).

Define the performance index for the sliding motion as follows:

$$J=\int_0^\infty (z^T z - r^2 w^T w)\,dt \tag{15}$$

where $r > 0$ is a design paramter to be determined to find the optimal guaranteed cost.

Definition 1. For a given positive constant r, the virtual control law $u_{vir} = Kx$ of Eq. (10) is a robust guaranteed cost control for the system of Eq. (14), if the following conditions holds for all the admissible parameter uncertainties $\underline{\Delta}$.
1. The sliding mode equation of Eq. (14) is asymptotically stable.
2. With the zero initial condition, the controlled output $z(t)$ satisfies

$$\|z\|_2 - r\|w\|_2 < 0 \tag{16}$$

where $\|.\|_2$ shows the standard norm in $L_2[0,+\infty)$.
3. The performance cost for the sliding mode is

$$J=\int_0^\infty (z^T z - r^2 w^T w)\,dt < J_{max}$$

Lemma 1. (Strict Bounded Real Lemma) The following statements are equivalent

1. There exists a matrix $Y > 0$ such that

$$A^T Y + YA + YBB^T Y + C^T C < 0$$

2. The Riccati equation

$$A^T X + XA + XBB^T X + C^T C = 0$$

has a stabilizing solution $X \geq 0$. Furthermore, if these statements hold then $X < Y$.

Theorem 1. For a given constant $r > 0$ and the performance cost of Eq. (15), a sufficient condition for the existence of guaranteed cost control $u_{vir} = Kx$ for system of Eq. (14) is that there exists a symmetric positive definite matrix X such that for all admissible parameter uncertainties $\underline{\Delta}$, the following inequality holds

$$\left(A + B_1 K + B_2 \underline{\Delta}(C + DK)\right)^T X + X\left(A + B_1 K + B_2 \underline{\Delta}(C + DK)\right)$$
$$+C^T C + K^T D^T DK - r^2 (C + DK)^T \underline{\Delta}^2 (C + DK) < 0 \qquad (17)$$

Proof. If there exists positive definite matrix X such that inequality of Eq. (17) holds, while the corresponding sliding mode of Eq. (14) uses the virtual control law $u_{vir} = Kx$, we construct a Lyapunov function $V(t)$ as

$$V(t) = x^T(t) X x(t) \qquad (18)$$

then $V(t)$ is positive definite, the derivative of $V(t)$ is

$$\dot{V}(t) = x^T(t)(F^T X + XF) x(t) \qquad (19)$$

where

$$F = A + B_1 K + B_2 \underline{\Delta}(C + DK)$$

By inequality of Eq. (17), for all the admissible uncertainties $\underline{\Delta}$, we get

$$\dot{V}(t) < -x^T(t)[C^T C + K^T D^T DK - r^2 (C + DK)^T \underline{\Delta}^2 (C + DK)] x(t)$$
$$\leq -\lambda_{min}[C^T C + K^T D^T DK - r^2 (C + DK)^T \underline{\Delta}^2 (C + DK)] \|x\|^2 < 0 \qquad (20)$$

where $\lambda_{min}(\cdot)$ shows the minimum eigenvalue of matrix $(\cdot)$. The obtained sliding mode from inequality of Eq. (20) is asymptotically stable, and Integrating Eq. (20) from 0 to ∞ at both sides of inequality, and use the stability for the system, we get

$$J < -V(\infty) + V(0) = x^T(0)Xx(0) \tag{21}$$

This shows that $u_{vir} = Kx$ is a guaranteed cost control for system of Eq. (14) with cost bound matrix X for all admissible uncertainties. Furthermore, we have

$$\dot{V}(t) + x^T(t)[C^TC + K^TD^TDK - r^2(C+DK)^T\underline{\Delta}^2(C+DK)]x(t) < 0$$

By the zero initial condition, we get

$$\int_0^\infty z^T(t)z(t)\,dt - r^2\int_0^\infty w^T(t)w(t)\,dt < -V(\infty) \leq 0 \tag{22}$$

to the arbitrary $w \in L_2[0,+\infty)$, we get the inequality $\|z\|_2 - r\|w\|_2 < 0$, the proof is completed.

Now, we present the virtual control law in terms of the solution of algebraic Riccati equation.

Theorem 2. For the given constant $r > 0$ and system performance index J of Eq. (15), if there exists a positive definite matrix X such that the Riccati equation

$$A^TX + XA + C^TC - XB_1(D^TD)^{-1}B_1^TX + \frac{1}{r^2}XB_2B_2^TX = 0 \tag{23}$$

has X as a solution and consider the equivalent control gain

$$K = -(D^TD)^{-1}B_1^TX \tag{24}$$

Then there exists a matrix $Y > 0$ such that $X < Y$ and Eq. (24) is a guaranteed cost control for the sliding mode of Eq. (14) with cost matrix Y.

Proof. With the equivalent control gain of Eq. (24), inequality of Eq. (17) can be written as

$$A^TY + YA + C^TC - YB_1(D^TD)^{-1}B_1^TY + \frac{1}{r^2}YB_2B_2^TY$$

$$-\left(\frac{1}{r^2}B_2^TY - \underline{\Delta}\,(C+DK)\right)^T r^2\left(\frac{1}{r^2}B_2^TY - \underline{\Delta}(C+DK)\right) < 0 \tag{25}$$

The above inequality holds for all the uncertainties, if

$$A^TY + YA + C^TC - YB_1(D^TD)^{-1}B_1^TY + \frac{1}{r^2}YB_2B_2^TY < 0 \tag{26}$$

By lemma 1 (Strict Bounded Real Lemma), the above inequality of Eq. (26) holds, if and only if there exist a stabilizing solution $X < Y$ such that

$$A^TX + XA + C^TC - X\left(B_1(D^TD)^{-1}B_1^T - \frac{1}{r^2}B_2B_2^T\right)X = 0 \tag{27}$$

This completes the proof of the theorem.

The guaranteed cost bound given in Eq. (21) is used as a measure of the cost associated with a control strategy. Furthermore, it is desirable to construct a controller of Eq. (24) which minimizes the cost bound under the worst possible uncertainty input. Note that the stabilizing solution to Riccati equation (23) will be monotonically decreasing with $r > 0$. Hence for each value of uncertainty bound $\mu > 0$, the optimal value of $r > 0$ will be determined to be the largest value such that inequality (16) is satisfied. However, to facilitate computation, we use the following inequality to approximate the optimal value of r instead of Eq. (17).

$$C^TC + X\left(B_1(D^TD)B_1^T - \frac{1}{r^2}B_2B_2^T\right)X < 0 \tag{28}$$

Finally, k_4 coefficient in Eq. (9) is computed so that the achieved acceleration will match the commanded acceleration. The closed-loop transfer function from A_{LC} to A_L in Eq. (14), when $\omega = 0$, is

$$G_{CL}(j\omega = 0) = k_4\frac{V(Z_1M_2 - Z_2M_1)}{(k_2 + k_1V)(Z_1M_2 - Z_2M_1)} \tag{29}$$

To get unity control system gain, we set the Eq. (29) to unity and get

$$k_4 = \frac{k_2 + k_1V}{V} \tag{30}$$

Based on Eq. (24) and (30), the sliding surface coefficients are selected to provide an upper bound on a given performance cost and to guarantee the stability under unmatched uncertainties.

3.2 Sliding Mode Control Law design

Now, the control law enforcing sliding mode in the surface $s = 0$ is examined. The control law of the following form is considered

$$\dot{\delta}_c = -\psi_1 A_L - \psi_2 q - \psi_3 \dot{q} + \psi_4 A_{LC} + \psi_5 s \tag{31}$$

where $\psi_{1\sim3}$ are discontinuous state feedback gains to be determined. The additional term $\psi_5 s$ is included for robustness improvement. The Lyapunov function is chosen as $L = \frac{1}{2}s^2$, which implies $\dot{L} = s\dot{s}$. The existence condition for sliding mode is fulfilled if

$$s\dot{s} = \sum_{i=1}^{3}\left(f_i + \frac{k_i - \psi_i}{T}\right)x_i\, s + \sum_{i=4}^{5}\left(f_i - \frac{k_i - \psi_i}{T}\right)x_i\, s < 0 \tag{32}$$

The following result gives a condition for the discontinuous gains $\psi_{1\sim5}$ to make the system stable.

Theorem 3. The autopilot system in Eq. (31) is stable if the following conditions are met

$$\psi_i = \begin{cases} k_i + \hat{T}\hat{f}_i + \hat{T}g_i, & f_i x_i s > 0 \\ k_i + \hat{T}\hat{f}_i, & f_i x_i s = 0\,, \quad for\ \ i = 1\sim3 \\ k_i + \hat{T}\hat{f}_i - \hat{T}g_i, & f_i x_i s < 0 \end{cases}$$

$$\psi_i = \begin{cases} k_i - \hat{T}\hat{f}_i - \hat{T}g_i, & f_i x_i s > 0 \\ k_i - \hat{T}\hat{f}_i, & f_i x_i s = 0\,, \quad for\ \ i = 4\sim5 \\ k_i - \hat{T}\hat{f}_i + \hat{T}g_i, & f_i x_i s < 0 \end{cases} \tag{33}$$

with

$$g_i \geq gF_i + |g-1|\left|\hat{f}_i\right|, \qquad for\ \ i = 1\sim5 \tag{34}$$

where the estimate $\hat{T}$ of the delay T is taken as the geometric mean of the known bounds, $0 < T_{min} \leq T \leq T_{max}$,

$$\hat{T} = \sqrt{T_{max}T_{min}} \tag{35}$$

The $\hat{f}_i$ of the parameter f_i are taken so that the estimation error on f_i should be bounded by some known value F_i,

$$\left|f_i - \hat{f}_i\right| \le F_i \tag{36}$$

and $g \ge \sqrt{T_{max}/T_{min}}$ is constant for all time.

Proof. If $\psi_{i=1\sim3}$ are chosen as $k_i + \hat{T}\hat{f}_i \pm \hat{T}g_i$ in accordance with the sign of $x_i s$, then

$$\left(f_i + \frac{k_i - \psi_i}{T}\right)x_i s = \left(f_i - \frac{\hat{T}\hat{f}_i}{T}\right)x_i s - \frac{g_i\hat{T}}{T}|x_i s|, \quad for\ i = 1\sim3 \tag{37}$$

If $\psi_{i=4\sim5}$ are also chosen as $k_i - \hat{T}\hat{f}_i \mp \hat{T}g_i$ according to the sign of $x_i s$, then

$$\left(f_i - \frac{k_i - \psi_i}{T}\right)x_i s = \left(f_i - \frac{\hat{T}\hat{f}_i}{T}\right)x_i s - \frac{g_i\hat{T}}{T}|x_i s|, \quad for\ i = 4\sim5 \tag{38}$$

Since $f_i = \hat{f}_i + (f_i - \hat{f}_i)$, where $\left|f_i - \hat{f}_i\right| \le F_i$, this in turn lead to

$$\left(f_i - \frac{\hat{T}\hat{f}_i}{T}\right)x_i s - \frac{g_i\hat{T}}{T}|x_i s| \le \left(F_i + \hat{f}_i - \frac{\hat{T}\hat{f}_i}{T}\right)x_i s - \frac{g_i\hat{T}}{T}|x_i s| \tag{39}$$

Thus, by choosing g_i to be large enough,

$$g_i \ge gF_i + |g-1|\left|\hat{f}_i\right| \ge \frac{T}{\hat{T}}F_i + \left|\frac{T}{\hat{T}} - 1\right|\left|\hat{f}_i\right| \tag{40}$$

We can guarantee that

$$s\dot{s} = \sum_{i=1}^{3}\left(f_i + \frac{k_i - \psi_i}{T}\right)x_i\, s + \sum_{i=4}^{5}\left(f_i - \frac{k_i - \psi_i}{T}\right)x_i\, s < 0 \tag{41}$$

From the definition of V, s converges to zero.

Notice that the control law is actually not $\dot{\delta}_c$ but δ_c, so a smooth control law is obtained by low pass filtering $\dot{\delta}_c$ [10]. Although the $\dot{\delta}_c$ shows chattering phenomenon because of a switching function, the δ_c applied to the real plant does not show the chattering phenomenon.

4. Numerical Simulation

Consider the uncertain flight control system described by the state equations

$$\begin{bmatrix} \dot{x} \\ \dot{u}_{vir} \end{bmatrix} = \begin{bmatrix} A(1+\Delta_1) & B_1(1+\Delta_2) \\ 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} x \\ u_{vir} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{T} \end{bmatrix} u$$

$$w = \Delta_3 z$$

$$z = Cx + Du_{vir}$$

where T is an uncertain delay but with known bounds $0.5ms \leq T \leq 1.5ms$, Δ_i are matrices of uncertain parameters with the given bound $\|\Delta_i\|_\infty < \frac{1}{8}$. We wish to construct an optimal guaranteed cost sliding mode control for this system which minimizes the bound on the cost index

$$J = \int_0^\infty (z^T z - r^2 w^T w)\, dt$$

This uncertain system and cost function are transformed into the uncertainty pulled out form in Eq. (14) and (15) where

$$A = \begin{bmatrix} -2.9 & 8819.8 & 0 \\ 0 & 0 & 1 \\ 0.2 & -642.3 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1949.2 \\ 0 \\ -554.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -0.4 & 0 & 0 \\ 0 & 1102.5 & 0 \\ 0.2 & 0 & 0 \\ 0 & -80.3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 243.6 \\ -69.3 \end{bmatrix}$$

and the $|\underline{\Delta}_1| < 1$.

In order to construct the required optimal quadratic guaranteed cost control, we must find the value of the parameter $r > 0$ which minimizes the trace of the stabilizing solution to Riccati equation (23) while satisfying inequality (28). This Riccati equation has been found to have a positive-definite solution for $r > 1.21$. Figure 1-(a) shows cost derivative versus r and (b) shows H_∞ norm from w to z. In order to facilitate computation, we have used the cost derivative of Eq. (28) instead of H_∞ norm. From the Fig. 1, we can find out that the optimal value is well approximated. we determine $r^* = 1.31$. Corresponding to this value of r, we obtain the following stabilizing solution to Riccati equation (23):

$$X = \begin{bmatrix} 0.0000 & 0.0008 & 0.0001 \\ 0.0008 & 3.5808 & 0.0521 \\ 0.0001 & 0.0521 & 0.0022 \end{bmatrix} \times 10^4 > 0$$

The corresponding optimal value of the cost bound is $trace(X) = 3.5829 \times 10^4$. Also, equation of (24) gives the corresponding optimal quadratic guaranteed cost control matrix

$$K = [-0.0030 \quad -4.2639 \quad -0.1708] \tag{42}$$

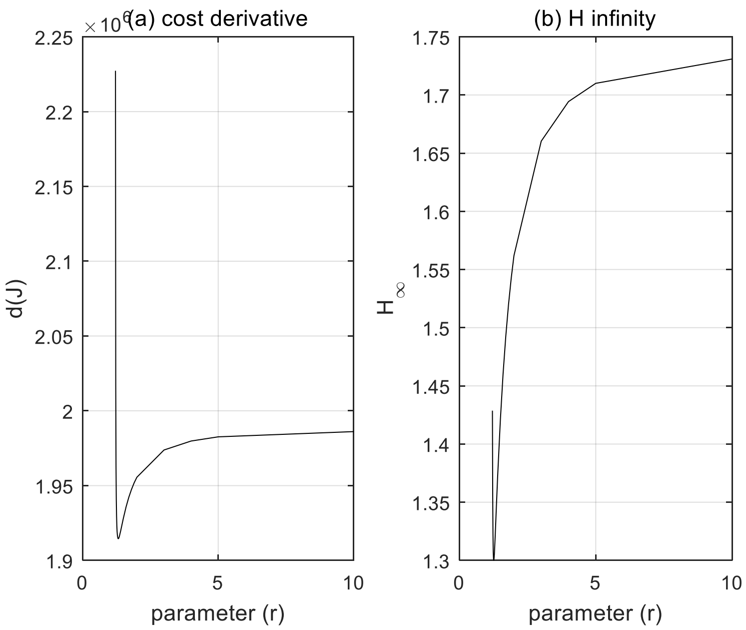

Fig. 1. Cost derivative & H_∞ versus r

Figure 2 and 3 show the response generated by the proposed guaranteed cost sliding mode controller. The switching surface used is

$$s = [K \quad 1]\begin{bmatrix} 1 \\ u_{vir} \end{bmatrix} - 0.0044A_{LC}$$

where the sliding surface gain K has been obtained in Eq. (42). The estimate of T is 0.87, and a value of 1.731 has been used for the upper limit of g. Figure 2 shows the corresponding step response, control rate commands $\dot{\delta}_c$, smooth control command δ_c, and sliding surface variable for a command $A_{LC} = 10g$. Although the $\dot{\delta}_c$ shows chattering phenomenon because of a switching function, the δ_c being applied to the real plant does not show the chattering phenomenon.

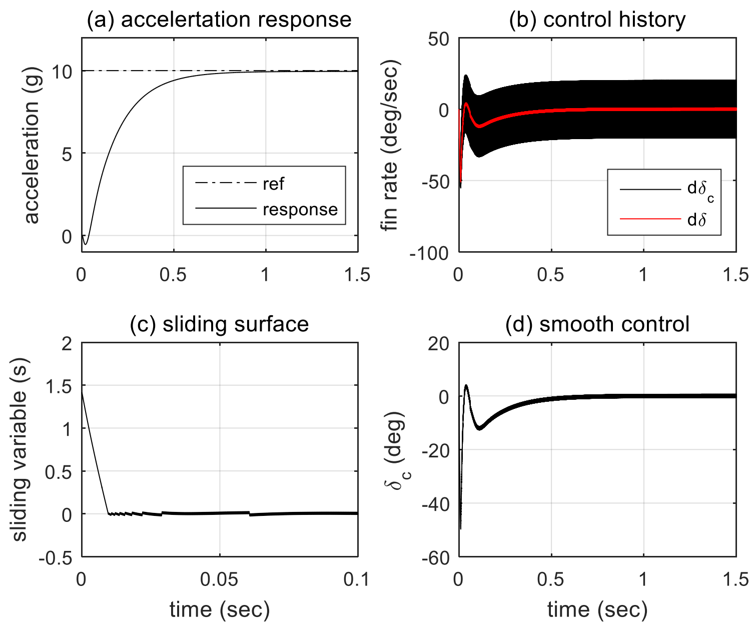

Fig. 2. Responses of the proposed controller

Figure 3 compares the robustness of the proposed and the conventional three-loop design [1] with respect to parameter uncertainties and four different unknown delays T =10ms, 50ms, 100ms and 200ms. The conventional controller starts to suffer badly as the unknown delay approaches the bandwidth of the system. However, the responses of the proposed controller keep insensitive to the uncertainty once sliding mode is established, and ensures the guaranteed cost of 3.5829×10^4 for all the admissible uncertainties.

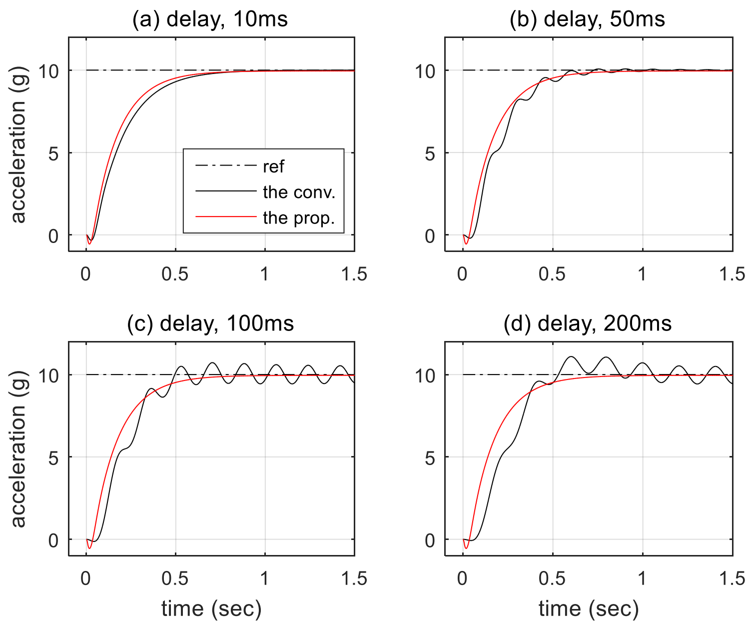

Fig. 3. Comparison of the proposed and conventional designs

5. Conclusion

If matching condition of uncertainty is not satisfied, robustness of sliding mode control becomes limited. In order to solve the unmatched problem, a sliding mode algorithm with a combination of an optimal guaranteed control method is presented. The proposed controller keeps the main advantages of standard sliding modes and an additional advantage that it can be used to provide an upper bound on a performance cost. Thus, system performance degradation incurred by unmatched uncertainties is guaranteed to be less than the given bound otherwise the siding motion may become unstable or performance may degrade. The simulation results show acceptable performance regardless of unmatched uncertainties. Based on its robust performance, the proposed controller can be considered as an efficient solution for controlling missile acceleration subject to unmatched uncertainty.

References

1. Hwang DH, Tahk MJ, "The Inverse optimal control problem for a three-loop missile autopilot," International Journal of Aeronautical & Space Sciences, vol. 19, no. 2, pp. 411-422, Apr. 2018.
2. Mracek CP and Ridgely DB, "Missile longitudinal autopilots: Connections between optimal control and classical topologies," AIAA guidance, navigation, and control conference, San Francisco, Aug. 2005.
3. Mracek CP and Ridgely DB, "Missile longitudinal autopilots: Comparison of multiple three loop topologies," AIAA guidance, navigation, and control conference, San Francisco, Aug. 2005.
4. Xu H, Mirmirani MD, and Ioannou PA, "Adaptive sliding mode control design for a hypersonic flight vehicle," Journal of guidance, control, and dynamics, vol. 27, no. 5, 2004, pp.829-839.
5. Shtessel, Buffington YJ, and Banda S, "Tailless aircraft flight control using multiple time scale reconfigurable sliding modes," IEEE Transactions on control system technology, vol. 10, no. 2, 2002, pp. 288-296.
6. Lee Y, Kim Y, Moon G, and Jun BE, "Sliding mode based attitude and acceleration controller for a velocity varying skid to turn missile," 13th European Control Conference (ECC2014), Curran Associates, Inc., Red Hook, NY, June 2014, pp.2951-2956.
7. Idan M, Shima T, and Golan OM, "Integrated sliding mode autopilot guidance for dual control missiles," Journal of guidance, control, and dynamics, vol. 30, no. 4, pp. 1081-1089, 2007.
8. Kim SH and Tahk MJ, "Missile acceleration controller design using proportional integral and non-linear dynamic control design method," Proceedings of Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering, vol. 255, no. 11, 2011. pp 882-897
9. Hwang DH and Tahk MJ, "Robustness improvement for a three-loop autopilot using discontinuous state feedback," International Journal of Aeronautical & Space Sciences, vol. 19, no. 3, pp. 661-674, Aug. 2018.
10. Park KB, Kim SW and Lee JJ, "Smooth variable structure controller for robot manipulator using virtual plant," Electronics Letters 23rd, vol. 31, no. 24, pp. 2134-2136, Nov. 1995
11. Petersen DS, and McFarlane DC, "Optimal guaranteed cost control and filtering for uncertain linear systems," IEEE Trans. on Automatic Control, vol. 39, no. 9, pp. 1971-1977, Sep. 1994.
12. Petersen DS, McFarlane DC, and Rotea MA, "Optimal guaranteed cost control of discrete-time uncertain linear systems," International Journal of Robust & Nonlinear Control, vol. 8, no. 7, pp. 649-657, Jul. 1998.
13. Petersen DS, "A stabilization algorithm for a class of uncertain systems," Systems & Control Letters, vol. 8, no. 3, pp. 181-188, 1987.
14. Yu L and Chu J, "An LMI approach to guaranteed cost control of linear uncertain time-delay systems," Automatica, vol. 35, no. 6, pp. 1155-1159, Jun. 1999.
15. Yu L, Chen G, and Yang M, "Optimal guaranteed cost control of linear uncertain systems: LMI approach," Proc. of the 14th IFAC World Congress, vol. G, pp. 541-546, 1999.
16. Vadim I. Utkin, Sliding modes in control and optimization, Springer-Verlag, Berlin Heidelberg New York.